

On numerical approaches to the analysis of topology of the phase space for dynamical integrability

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In this paper we consider the possibility to use numerical simulations for a computer assisted analysis of integrability of dynamical systems. We formulate a rather general method of recovering the obstruction to integrability for the systems with a small number of degrees of freedom. We generalize this method using the results of KAM theory and stochastic approaches to the families of parameter depending systems. This permits the localization of possible integrability regions in the parameter space. We give some examples of application of this approach to dynamical systems having mechanical origin.

Keywords: Integrability, numerical approach, phase space topology, KAM theory, Monte-Carlo method.

I. INTRODUCTION

This paper is the first in a series of works devoted to description of the possibilities of application of numerical methods to analysis of integrability of dynamical systems. The problem of integrability has been studied since the middle of the XIX century, when the question of primary interest was to be able to integrate the system of differential equations by inversion of functions and quadratures, that is give a more or less explicite solution. Nowadays with the development of methods relating symplectic geometry and dynamics the notion is extended, namely one studies the existence of an appropriate number of conserved quantities (first integrals, invariant measure, etc), possessing some properties.

In this paper we will consider integrability in the Liouville-Arnold sense, namely for a hamiltonian system with n degrees of freedom it is the existence of n independent first integrals in involution ([1]). In particular we will pay attention to the independence condition from the topological point of view. We propose a constructive method of computer assisted proof of non-integrability of systems with small dimensional configuration space based on the analysis of the topology of the phase space. We generalize it using the results of the Kolmogorov-Arnold-Moser theory to the systems with parameters. We also provide examples of application of this method to some dynamical systems having mechanical origin.

II. METHOD OF SECTIONS

Consider an autonomous hamiltonian system with a two-dimensional configuration space Q , its phase space T^*Q is of dimension 4. Since the right hand sides of the equations of motion do not depend explicitly on time, any trajectory of this systems belongs to a constant hamiltonian (energy) level hypersurface having the dimension 3. For complete integrability of the system with two degrees of freedom the existence of another first integral independent with energy integral is needed – in this case any trajectory would belong to the 2-dimension manifold defined by the intersection of the level surfaces of these first integrals.

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A. Idea of the method

This simple topological consideration alone does not give a method of analysis of integrability, but together with a good visualization algorithm it permits to give an answer for a rather large class of systems. Effective visualization of the dynamics in a 4-dimensional phase space is not an easy task, therefore we consider the intersection of a trajectory of the system with 2-dimensional planes in it. Let us discuss the possible results of this intersection.

The dimension of intersection of two generic manifolds of dimensions n_1 and n_2 in an N -dimensional space is given ([2]) by the equation

$$\dim = n_1 + n_2 - N. \quad (1)$$

In our case the manifold swept by the trajectory intersects a 2-dimensional plane ($n_2 = 2$) in the phase space of dimension $N = 4$. Energy conservation guarantees that the dimension (n_1) of this manifold is at most 3. Thus we can observe two possible cases depending on the existence of additional first integral: either $n_1 = 2$ and the intersection is 0-dimensional (finite set of points), or $n_1 = 3$ and the intersection is of dimension 1 (finite set of curves). It is clear that the presence of curves in the intersection is an obstruction to complete integrability. Absence of curves however does not directly mean integrability, since it can be the consequence of various reasons – we will discuss them more precisely in the section III in the context of generalization of this method.

Let us be more concrete on this idea about intersection and prove in the particular case the formula (1). Choose the canonical coordinates q_1, p_1, q_2, p_2 of the phase space T^*Q (assume for simplicity that it can be done globally). Then an arbitrary two-dimensional plane (or, better to say two-dimensional linear subspace) is given by the system of equations

$$\begin{aligned} a_1 q_1 + b_1 p_1 + c_1 q_2 + d_1 p_2 &= e_1, \\ a_2 q_1 + b_2 p_1 + c_2 q_2 + d_2 p_2 &= e_2. \end{aligned} \quad (2)$$

for some constants $a_i, b_i, c_i, d_i, e_i, i = 1, 2$. Since the system is autonomous any trajectory belongs to the energy level

$$I_1 \equiv H = h. \quad (3)$$

The conditions (2, 3) on 4 coordinates in the phase space define a 1-dimensional manifold. If there exists another first integral

$$I_2 = \text{const} \quad (4)$$

the system (2, 3, 4), if it is compatible, admits a finite set of solutions. It is now easy to see that it is important to pay attention to the independence of the conditions (2) from (3) and (4). This is not difficult to guarantee for the energy integral, since it is non-linear and usually known; if the other first integral exists and the equation (4) turns out to be dependent with (2) it means that the system admits some degeneracy, that will be seen in the numerical simulation. To avoid “false detection” by this method it is enough to consider the intersection with a couple of mutually independent planes, that is done in practice.

B. Triple pendulum

As one of the main examples of application of the method in this work we will consider the flat motion of the pendulum-type systems. A multiple pendulum is the system of point masses, connected by weightless inextensible rods, the first of the points being fixed. For a triple pendulum these conditions read

$$(\mathbf{r}_i - \mathbf{r}_{i-1})^2 = l_i^2 \quad i = 1, 2, 3,$$

where \mathbf{r}_i corresponds to the i -th mass, the fixed point is \mathbf{r}_0 . One can also consider a more general case of constraints given by arbitrary polynomials of degree 2, which however does not always correspond to a physical configuration.

The choice of this system is motivated by several factors. Let us note that the case of a double pendulum is well studied. In particular its free motion on the plane is a classical example of a completely integrable system

(see for example [3]). In the notation of the previous section taking the angular momentum for I_2 one obtains two independent conditions (3, 4), having a 0-dimensional intersection with (2). But already in the presence of gravity the non-integrability can be shown and some chaotic behaviour has been observed, that is the trajectory is rather dense on the energy level-surface, that can be seen on the sections. The problem of control has also been studied for this system ([4]), among interesting results one can mention that a double pendulum can be stabilized in the upright position by controlling only one degree of freedom.

Meanwhile the dynamics of the triple pendulum is almost not studied, although it is interesting for applications. We have shown rather directly (without going much into details about topology, but writing explicitly the coordinates) that the behaviour of the system is rather irregular and it is not integrable [17]. This results can be useful to study the dynamics of a multiple lattice, being one of the low-temperature models of the polymer molecule. The idea of describing the interaction of subunits in molecular dynamics by lagrangian mechanics extends the harmonic model used in [5]. Actually the study of thermodynamic properties of the pendulum systems in the framework of the model from [5] explains the author's interest to their integrability.

We will not describe this example in full details since we have already done it partially in [6] and in [7]. Let us just note that in the original setting the system obviously has 3 degrees of freedom, but since it admits the angular momentum first integral it can be reduced by the Routh transform ([8]) to the one having the 4-dimensional phase space, that is precisely the range of application of the method of sections. Let us also note, that this reduction is profitable to understand the topology of the phase space, but to perform numerical simulations it is more convenient to use the lagrange multipliers ([9]) and then compute the reduced coordinates from the cartesian ones.

A typical result of the numerical simulations is shown on fig. 1. The coordinates in the reduced phase space are the angles β_1, β_2 between the segments of the pendulum and their derivatives. In the hamiltonian formalism one should actually use the momenta instead of velocities, but we use the natural duality of T^*Q and TQ , so it gives an equivalent picture from the topological point of view. The intersecting planes are chosen to be parallel to the coordinate ones. The presence of curves in the intersection clearly shows non-integrability.

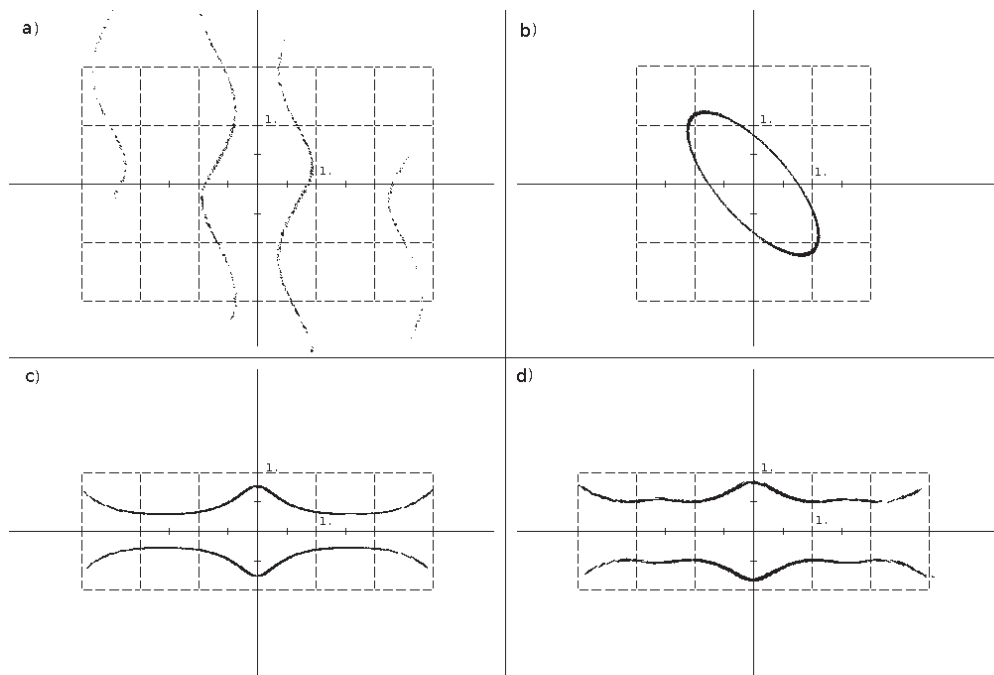


FIG. 1: The intersection of the trajectory of the triple pendulum with the planes: a). $(\beta_1 = 1, \beta_2 = 1)$, b). (β_1, β_2) , c). (β_1, β_1) , d). (β_2, β_2) .

C. Satellite dynamics

Let us consider another mechanical example for which the method is applicable: the motion of the dynamically symmetric satellite on the circular orbit. In the orbital coordinate system, the axes of which are directed along the radius of the orbit of the center of masses of the satellite, its normal and its binormal, the position of the satellite is given by the three Euler angles ψ, θ, φ . Denoting the corresponding momenta by $p_\psi, p_\theta, p_\varphi$, one obtains the following hamiltonian function:

$$H = \frac{p_\psi^2}{2 \sin^2 \theta} + \frac{p_\theta^2}{2} - p_\psi \operatorname{ctg}(\theta) \cos(\psi) - \alpha \beta p_\psi \frac{\cos \theta}{\sin^2 \theta} - p_\theta \sin(\psi) + \alpha \beta \frac{\cos \psi}{\sin \theta} + \frac{\alpha^2 \beta^2}{2 \sin^2 \theta} + \frac{3}{2}(\alpha - 1) \cos^2 \theta, \quad (5)$$

where $\alpha = C/A$; A, B, C are the principal moments of inertia ($A = B$). The coordinate φ is cyclic, thus we can fix the corresponding momenta $p_\varphi = \alpha \beta = \text{const}$, where β denotes the ratio of the orbital angular velocity and the projection of the angular velocity of the satellite to its symmetry axis. Let us consider the values of $\alpha = 4/3, \beta = 0$, for which the stability of equilibrium solutions is studied in [10]. The reduced system has two degrees of freedom and is described by the hamiltonian:

$$H = \frac{p_\psi^2}{2 \sin^2 \theta} + \frac{p_\theta^2}{2} - p_\psi + \frac{1}{2} \sin^2 \psi \sin^2 \theta.$$

That is we can again apply the method of sections. The intersections that one obtains (fig. 2) show its non-integrability.

Let us however note, that to obtain the sections with the curves for this problem we had to analyse a lot of different initial conditions. This indicates that the system can be *locally integrable*, that is possess an additional first integral for a certain subset of initial data.

D. Details of the method of sections

Let us now discuss the possibilities of application of the method of sections to other systems, in particular let us comment on the class of systems for which it can result in the precise conclusion on integrability. As we have noted in the very beginning the natural restriction comes from the dimension of the phase space, or more precisely from the possibility to reduce it. This however covers a large class of systems for which the integrability is an open question, such as the dynamics of the triple lattice, the motion of a mass point in the symmetric potential in a 3-dimensional space, dynamics of geodesics on the curved surfaces etc.

The only difficulty is the choice of convenient coordinates for constructing the sections. But this problem can be naturally solved when the system admits the angle-parametrization like in the described examples. Note that such systems often arise in the applications. According to the theorem by V.V.Kozlov ([11]) it is interesting to consider the systems with the genus of the configuration space at most 1, since if it is not the case one knows that the system is not analytically integrable; but it means that basically any choice of coordinates has the structure of angles. We can however study more general surfaces if we are interested in a larger class of first integrals. Then it is good to make sure that the motion takes place in a bounded domain of the phase space, to accumulate a sufficiently representative intersection. This condition is also rather natural at least for the systems with compact configuration space.

Let us also note a technical difficulty of the method. First, to conclude non-integrability one needs to compute a rather long trajectory, since the intersection of the trajectory with a surface of small dimension is a rear event in contrast, say, to Poincaré sections where every loop gives a point in the plane. But this reduction of the dimension is done intensionally, as it permits more effectively classify different cases: it is much easier to distinguish points from curves than curves from thin domains that can occur in the Poincaré sections. Our method also permits not to pay much attention to transversality of the trajectory to the planes. All this only results in the need to apply reliable algorithms of numerical integration.

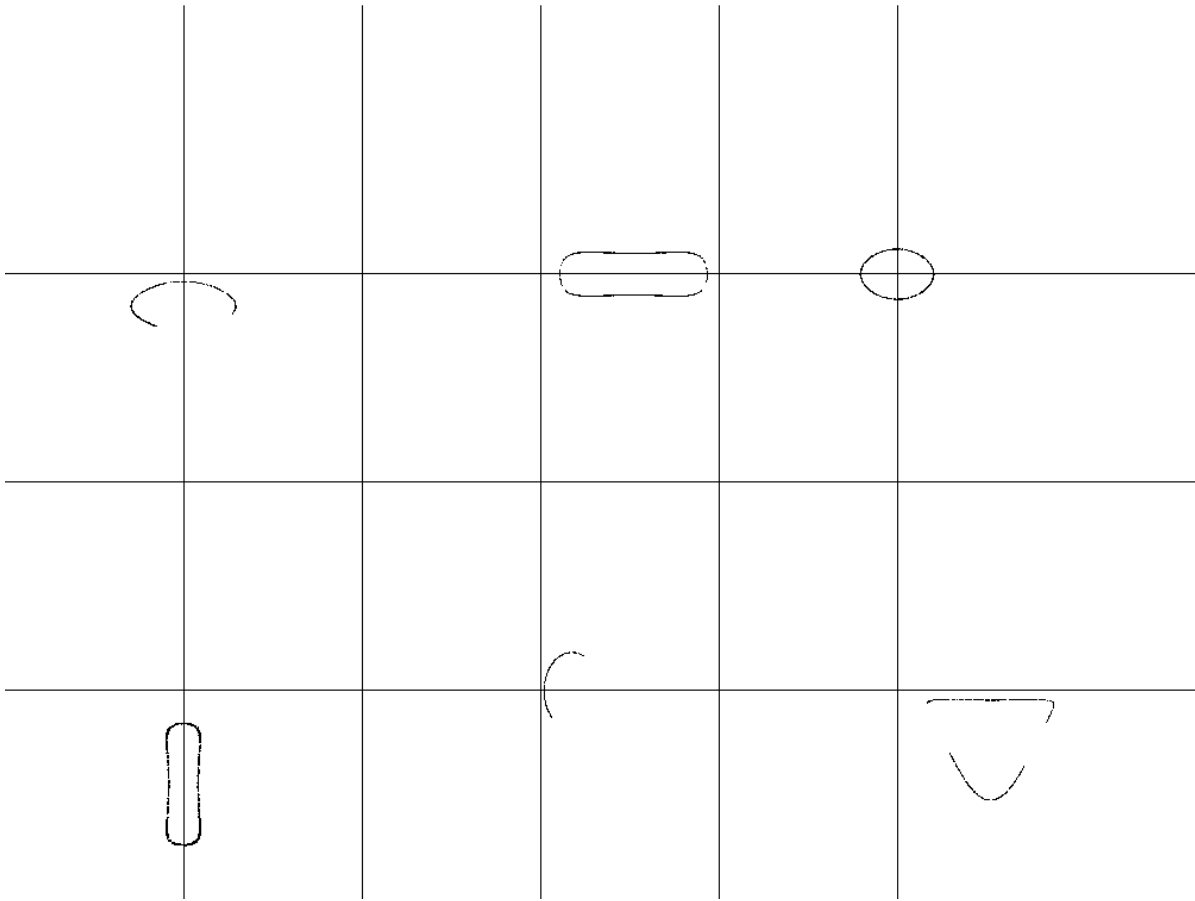


FIG. 2: The intersection of the trajectory with the planes, parallel to the coordinate ones, containing the point $\psi = 0.28, \theta = 0.82, p_\psi = 0.15, p_\theta = 0.37$.

Another thing that we haven't yet discussed in details is the interpretation of the “empty” (0-dimensional) intersections. The existence of such sections does not permit to make any direct conclusion since it only means that a given trajectory is by chance more regular. But in the next section we will explain the origin of such sections for non-integrable systems and discuss the possibility to use them to qualitatively describe the behaviour of the systems.

III. GENERALIZATION OF THE METHOD OF SECTIONS VIA THE KAM THEORY

The method of sections as we have seen permits to prove non-integrability for a given dynamical system. This problem however is rather special, more often one wants to find the relation between the parameters when integrability is possible. In this section we will consider an example of a two parameter system showing how one can extend the method of section to this problem. Despite the discrete nature of the method it permits (via some extra mathematical considerations) also to draw conclusions for a continuous range of the parameters.

A. Pendulum-type systems

Consider free flat motion of a system obtained from the triple pendulum by moving the fixing points of the next segment along the previous one (fig. 3). We shall call such objects pendulum-type system. In particular they describe the motion of a physical pendulum, i.e. rigid bodies fixed between themselves.

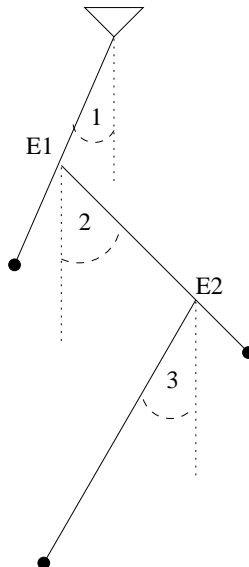


FIG. 3: A pendulum type system parametrized by the angles. The distances: $E1 = \varepsilon_1 l_1$, $E2 = \varepsilon_2 l_2$

The configuration of this system is described by two dimensional vectors \mathbf{r}_i and velocities \mathbf{v}_i ($i = 1, 2, 3$) with the following constraints:

$$\begin{aligned}\varphi_1 &= (\mathbf{r}_1 - \mathbf{r}_0)^2 - l_1^2 = 0, \\ \varphi_2 &= (\mathbf{r}_2 - \varepsilon_1 \mathbf{r}_1)^2 - l_2^2 = 0, \\ \varphi_3 &= (\mathbf{r}_3 - (\varepsilon_1 \mathbf{r}_1 + \varepsilon_2 (\mathbf{r}_2 - \varepsilon_1 \mathbf{r}_1)))^2 - l_3^2 = 0,\end{aligned}\tag{6}$$

the point \mathbf{r}_0 is fixed. The parameters ε_i characterize linearly the fixing point of the $(i+1)$ -st segment to the i -th one: $\varepsilon_i = 1$ corresponds to fixing at the endpoint (like in the multiple pendulum), $\varepsilon_i = 0$ – to the fixed point (like two non-interacting pendula with the same fixed point).

Integrability. Note that for $\varepsilon_1 = \varepsilon_2 = 1$ we obtain the non-integrable case of the triple pendulum studied before. For $\varepsilon_1 = 0$ and any ε_2 the system decouples to non-interacting simple pendulum and a double pendulum-type system, and becomes obviously integrable. And in the particular case when both $\varepsilon_1 = \varepsilon_2 = 0$, it decouples to three non-interacting simple pendula. For other values of ε_i we will be again interested in the existence of a sufficient number of independent first integrals.

B. Topology of the phase space

For all values of $\varepsilon_1, \varepsilon_2$ the system is invariant under rotation around the fixed point \mathbf{r}_0 , thus it possesses the Nöther integral of angular momentum. Since the position is still characterized by the three angles (fig. 3), this integral permits to perform Routh reduction to the 4-dimensional phase space. To make this reduction explicitly we again study the dynamics in terms of the angles between the segments of the pendulum-type system $\beta_i = \alpha_{i+1} - \alpha_i$, $i = 1, 2$ and their derivatives. Now for any fixed couple of $(\varepsilon_1, \varepsilon_2)$, one we can apply the method of sections.

Typical results for the pendulum-type systems are represented on figs. 4(a)–4(d), showing the intersection with the planes $(\dot{\beta}_2 = 0, \beta_2 = 0)$, $(\dot{\beta}_2 = 0, \beta_2 = 0)$, $(\dot{\beta}_1 = 0, \beta_2 = 0)$ $(\dot{\beta}_2 = 1, \beta_1 = 1)$, $(\beta_2 = 0, \beta_1 = 0)$, $(\dot{\beta}_1 = 0, \beta_1 = 0)$

This figures show that the sections contain curves, it means the corresponding systems do not admit an additional first integral. The similar figures have been obtained (i.e. initial conditions found – [18]) for all couples $\varepsilon_1, \varepsilon_2 \in [0, 1] \times [0, 1]$ with the step 0.02, except the trivial ones ($\varepsilon_1 = 0$), that means non-integrability of the corresponding systems.

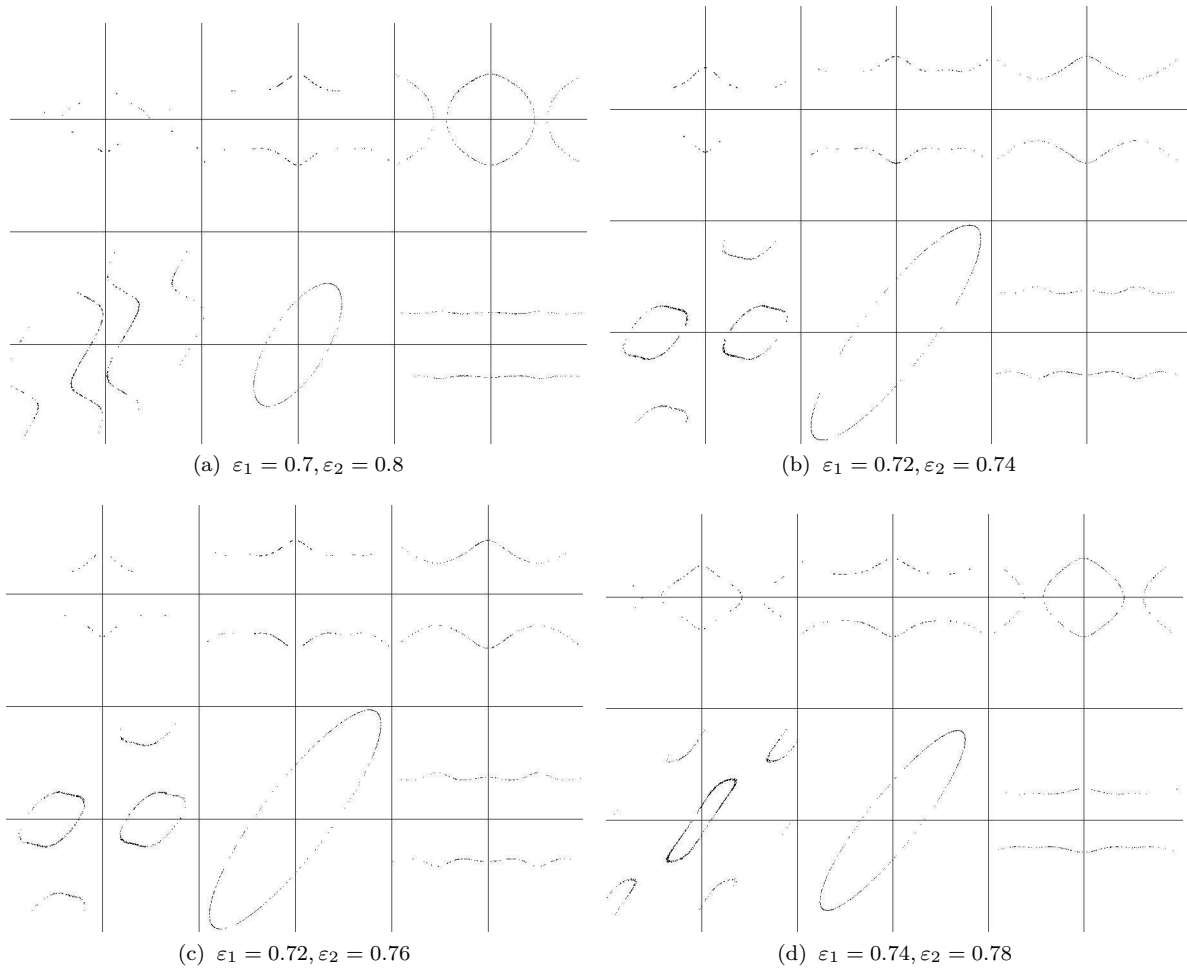


FIG. 4: Sections for various values of $\varepsilon_1, \varepsilon_2$

C. Generalization to continuous parameters

Note that the sections for close values of parameters and close initial data look rather similar (Fig. 4(b), 4(c)). One thus is tempted to conjecture that the systems between the two values of parameters are also non-integrable. Alone this statement is certainly false, but we can make sense out of it by some extra consideration. Let us make it more precise.

The results of Poincaré ([12, 13]) show that the systems obtained from integrable one by a perturbation usually fail to be integrable. However according to the Kolmogorov-Arnold-Moser theorem (see for example [1]) for the systems close to integrable a set of positive measure of invariant tori preserves its topology. In our terms it means, that when the initial point even of a non-integrable system lies on such a torus the sections do not contain curves. And when we approach an integrable system in the space of parameters the probability to be on such a torus increases.

This effect is indeed observed in the neighborhood of the integrable case ($\varepsilon_1 = 0$) for arbitrary chosen initial points. The figure 5(a) shows the distribution obtained by the Monte-Carlo method ([14]) of such tori, i.e. the proportion of “empty” sections if one starts from a pseudo-random point in the phase space for the given values of parameters ($\varepsilon_1, \varepsilon_2$). The figure 5(b) shows the same distribution only depending on ε_1 . One sees from them, that except the segment $\varepsilon_1 = 0$, in the square $(\varepsilon_1, \varepsilon_2) \in [0, 1] \times [0, 1]$ there are no pronounced maxima, that shows non-integrability of all the systems with parameters different from $\varepsilon_1 = 0$. That is in the family of triple pendulum-type systems the only integrable situation corresponds to decoupling the systems to non-interacting subsystems of smaller dimensions.

Generalization of the method of sections. It is clear that the same approach is not limited to the above example. We can apply it to any family of systems with arbitrary number of parameters the phase space of which

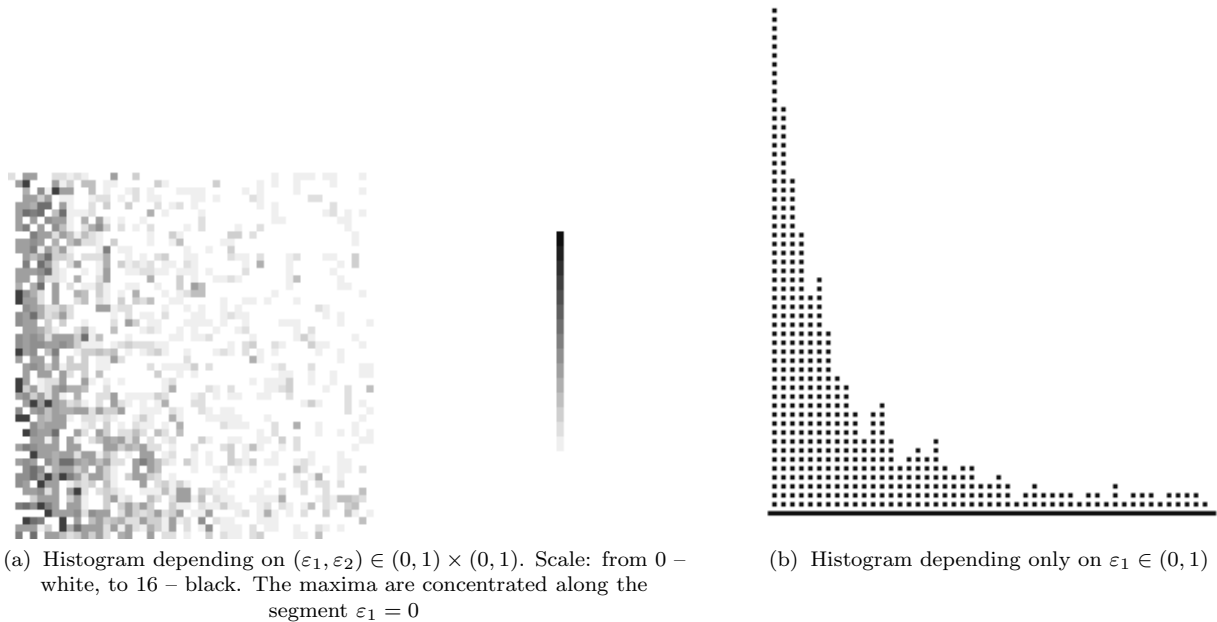


FIG. 5: Distributions of the Liouville tori preserving their topology

can be uniformly reduced to the 4-dimensional. One needs to choose a domain D in the parameter space, apply the method of sections to a sufficient number of pseudo-random initial points for all values of parameters from a rather dense set in D and compute the estimation of the probability to start with an invariant torus its preserving the topology. The subdomains of D corresponding to maxima of the obtained distribution are the candidates for integrability.

IV. CONCLUSION

In this paper we have described the topological properties of the phase space of the small dimensional dynamical systems in the form convenient for verification. We have proposed a method of computer assisted proof of non-integrability for a given dynamical system and generalized it to the continuous range of parameters. This is a natural application of the classical Monte-Carlo method, permitting by a well-developed technique, using the pseudo-random quantities construct a qualitative picture of the behaviour of a deterministic system. Together with the results of the Kolmogorov-Arnold-Moser theory it permits to answer rather complicated questions on integrability. Such statistical approaches are often used in quantum mechanics and in various inverse problems of image recognition.

As we have mentioned in the beginning of this paper there are other ways to study dynamical integrability using numerical methods. For example the meromorphic non-integrability of the examples discussed in the section II can be proved by constructing the monodromy group ([15]) and application of the results of ([16]). One of the main motivations to develop these methods for us is studying more involved problems like systems with delay naturally appearing in biological modeling or relativistic celestial mechanics.

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